CT-TN-2024-10 **蔡承颖、唐家振**
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关于 Eq. (4.40) 的推导

根据赵老师教科书的 Eq. (4.36)，待解的方程有

$$\begin{aligned} y_1'' + k_\beta^2 y_1 &= 0 \\ y_2'' + k_\beta^2 y_2 &= \frac{N r_0 W_0}{2\gamma C} y_1 \end{aligned} \tag{1}$$

预期结果 Eq. (4.38) 为

$$\tilde{y}_1(s) = \tilde{y}_1(0) e^{-ik_\beta s} \tag{2}$$

其中，

$$\tilde{y}_1 = y_1 + \frac{i}{k_\beta} y_1' \tag{3}$$

与 Eq. (4.40)

$$\tilde{y}_2(s) = \tilde{y}_2(0) e^{-ik_\beta s} + i\frac{N r_0 W_0}{4\gamma C k_\beta} \left[\frac{1}{k_\beta} \tilde{y}_1^*(0) \sin k_\beta s + \tilde{y}_1(0) s e^{-ik_\beta s} \right] \tag{4}$$

以下试着给出推导。在开始前，先总结常微分方程的一个结果。对非齐次二阶微分方程，形如 Eq. (1) 的第二式，驱动项为 sin 或 cos 函数形式时，完整解 y 可写成 $y = y_h + y_p$，其中，齐次解 y_h 有

$$y_h(s) = A \cos k_\beta s + \frac{B}{k_\beta} \sin k_\beta s \tag{5}$$

非齐次解 (或特解)y_p 有

$$y_p(s) = C_1 s \cos k_\beta s + C_2 s \sin_\beta s \tag{6}$$

1 $y_1(s)$ 解

现在，先从 y_1 方程开始，由于没有驱动项，仅有齐次解

$$y_1(s) = y_1(0) \cos k_\beta s + \frac{y_1'(0)}{k_\beta} \sin k_\beta s \tag{7}$$

为了表示成 $\tilde{y}_1(s)$ 形式，还需要 $y_1'(s)$，有

$$y_1'(s) = -k_\beta y_1(0) \sin k_\beta s + y_1'(0) \cos k_\beta s \tag{8}$$

于是，

$$
\begin{aligned}
\tilde{y}_1(s) &= y_1(s) + \frac{i}{k_\beta} y_1'(s) \\
&= y_1(0)\cos k_\beta s + \frac{y_1'(0)}{k_\beta}\sin k_\beta s - i y_1(0)\sin k_\beta s + \frac{i}{k_\beta} y_1'(0)\cos k_\beta s \\
&= \left(y_1(0) + \frac{i}{k_\beta} y_1'(0)\right)\cos k_\beta s - i\left(y_1(0) + \frac{i}{k_\beta} y_1'(0)\right)\sin k_\beta s \\
&= \left(y_1(0) + \frac{i}{k_\beta} y_1'(0)\right)(\cos k_\beta s - i\sin k_\beta s) \\
&= \tilde{y}_1(0) e^{-ik_\beta s}
\end{aligned} \tag{9}
$$

此即 Eq. (4.38)。

2 $y_2(s)$ 解

由于 y_2 方程存在驱动项，完整解为

$$
y_2(s) = A\cos k_\beta s + \frac{B}{k_\beta}\sin k_\beta s + C_1 s\cos k_\beta s + C_2 s \sin_\beta s \tag{10}
$$

其中，系数 A, B, C_1, C_2 待求。如果将 $s = 0$ 处的初始条件写成 $y_2(0)$ 与 $y_2'(0)$，则根据上式，有

$$
\boxed{y_2(0) = A} \tag{11}
$$

现在，将 Eq. (10) 对 s 微分，

$$
y_2'(s) = -k_\beta A\sin k_\beta s + B\cos k_\beta s + C_1\cos k_\beta s - C_1 k_\beta s\sin k_\beta s + C_2\sin k_\beta s + C_2 k_\beta s\cos k_\beta s \tag{12}
$$

则

$$
\boxed{y_2'(0) = B + C_1} \tag{13}
$$

再来，将 Eq. (12) 再对 s 微分，有

$$
\begin{aligned}
y_2''(s) = &-k_\beta^2 A\cos k_\beta s - Bk_\beta\sin k_\beta s - k_\beta C_1\sin k_\beta s - k_\beta C_1\sin k_\beta s - k_\beta^2 C_1 s\cos k_\beta s \\
&+ k_\beta C_2\cos k_\beta s + C_2 k_\beta\cos k_\beta s - k_\beta^2 C_2 s\sin k_\beta s
\end{aligned} \tag{14}
$$

则

$$
y_2''(0) = -k_\beta^2 A + 2k_\beta C_2 \tag{15}
$$

其中，C_2 可以通过 Eq. (1) 推知。留意，这里没有出现 C_1。如何决定 C_2? 将 Eq. (7) 代入 Eq. (1) 的第二式，考虑 $s = 0$ 情况，有

$$
y_2''(0) + k_\beta^2 y_2(0) \stackrel{!}{=} \frac{Nr_0W_0}{2\gamma C} y_1(0) \tag{16}
$$

比较系数后，有

$$2k_\beta C_2 = \frac{Nr_0W_0}{2\gamma C}y_1(0) \quad \Rightarrow \quad \boxed{C_2 = \frac{Nr_0W_0}{4\gamma Ck_\beta}y_1(0)} \tag{17}$$

还剩 C_1，如何决定?

按类似思路，将 Eq. (14) 再对 s 微分，有

$$\begin{aligned} y_2'''(s) = & k_\beta^3 A\sin k_\beta s - Bk_\beta^2\cos k_\beta s - k_\beta^2 C_1\cos k_\beta s - k_\beta^2 C_1\cos k_\beta s - k_\beta^2 C_1\cos k_\beta s \\ & + k_\beta^3 C_1 s\sin k_\beta s - k_\beta^2 C_2\sin k_\beta s - C_2 k_\beta^2\sin k_\beta s - k_\beta^2 C_2\sin k_\beta s - k_\beta^3 C_2 s\cos k_\beta s \end{aligned} \tag{18}$$

则

$$y'''(0) = -Bk_\beta^2 - 3C_1k_\beta^2 \tag{19}$$

还是回到 Eq. (1) 的第二式，考虑 $s=0$ 情况，但现在等式两边再微分一次

$$y_2'''(0) + k_\beta^2 y_2'(0) = \frac{Nr_0W_0}{2\gamma C}y_1'(0) \tag{20}$$

比较系数后，有

$$-Bk_\beta^2 - 3C_1k_\beta^2 + k_\beta^2(B+C_1) = \frac{Nr_0W_0}{2\gamma C}y_1'(0) \quad \Rightarrow \quad \boxed{C_1 = -\frac{Nr_0W_0}{4\gamma Ck_\beta}\frac{y_1'(0)}{k_\beta}} \tag{21}$$

于是，$y_2(s)$ 的完整解为

$$\begin{aligned} y_2(s) = & y_2(0)\cos k_\beta s + \frac{1}{k_\beta}\left[y_2'(0) + \frac{Nr_0W_0}{4\gamma Ck_\beta}\frac{y_1'(0)}{k_\beta}\right]\sin k_\beta s \\ & - \frac{Nr_0W_0}{4\gamma Ck_\beta}\frac{y_1'(0)}{k_\beta}s\cos k_\beta s + \frac{Nr_0W_0}{4\gamma Ck_\beta}y_1(0)s\sin k_\beta s \end{aligned} \tag{22}$$

为了表示成 $\tilde{y}_2(s)$ 形式，还需要 $y_2'(s)$，有

$$\begin{aligned} y_2'(s) = & -k_\beta y_2(0)\sin k_\beta s + \left[y_2'(0) + \frac{Nr_0W_0}{4\gamma Ck_\beta}\frac{y_1'(0)}{k_\beta}\right]\cos k_\beta s - \frac{Nr_0W_0}{4\gamma Ck_\beta}\frac{y_1'(0)}{k_\beta}\cos k_\beta s \\ & + \frac{Nr_0W_0}{4\gamma Ck_\beta}y_1'(0)s\sin k_\beta s + \frac{Nr_0W_0}{4\gamma Ck_\beta}y_1(0)sk_\beta\cos k_\beta s + \frac{Nr_0W_0}{4\gamma Ck_\beta}y_1(0)\sin k_\beta s \end{aligned} \tag{23}$$

于是，

$$
\begin{aligned}
\tilde{y}_2(s) &= y_2(s) + \frac{i}{k_\beta} y_2'(s) \\
&= y_2(0)\cos k_\beta s + \frac{1}{k_\beta}\left[y_2'(0) + \frac{Nr_0W_0}{4\gamma C k_\beta}\frac{y_1'(0)}{k_\beta}\right]\sin k_\beta s - \frac{Nr_0W_0}{4\gamma C k_\beta}\frac{y_1'(0)}{k_\beta} s\cos k_\beta s \\
&\quad + \frac{Nr_0W_0}{4\gamma C k_\beta} y_1(0) s \sin k_\beta s - i y_2(0)\sin k_\beta s + \frac{i}{k_\beta}\left[y_2'(0) + \frac{Nr_0W_0}{4\gamma C k_\beta}\frac{y_1'(0)}{k_\beta}\right]\cos k_\beta s \\
&\quad - \frac{i}{k_\beta}\frac{Nr_0W_0}{4\gamma C k_\beta}\frac{y_1'(0)}{k_\beta}\cos k_\beta s + \frac{i}{k_\beta}\frac{Nr_0W_0}{4\gamma C k_\beta} y_1'(0) s \sin k_\beta s \\
&\quad + i\frac{Nr_0W_0}{4\gamma C k_\beta} y_1(0) s\cos k_\beta s + \frac{i}{k_\beta}\frac{Nr_0W_0}{4\gamma C k_\beta} y_1(0)\sin k_\beta s \\
&= y_2(0)\cos k_\beta s - i y_2(0)\sin k_\beta s + \frac{1}{k_\beta}\left[y_2'(0) + \frac{Nr_0W_0}{4\gamma C k_\beta}\frac{y_1'(0)}{k_\beta}\right]\sin k_\beta s \\
&\quad + \frac{i}{k_\beta}\left[y_2'(0) + \frac{Nr_0W_0}{4\gamma C k_\beta}\frac{y_1'(0)}{k_\beta}\right]\cos k_\beta s \\
&\quad + i\frac{Nr_0W_0}{4\gamma C k_\beta}\left(-i y_1(0) s\sin k_\beta s + i\frac{y_1'(0)}{k_\beta} s\cos k_\beta s + y_1(0) s\cos k_\beta s\right. \\
&\qquad\qquad \left. + \frac{1}{k_\beta} y_1(0)\sin k_\beta s - \frac{1}{k_\beta}\frac{y_1'(0)}{k_\beta}\cos k_\beta s + \frac{1}{k_\beta} y_1'(0) s\sin k_\beta s\right) \\
&= \tilde{y}_2(0)\cos k_\beta s - i\tilde{y}_2(0)\sin k_\beta s + i\frac{Nr_0W_0}{4\gamma C k_\beta}\frac{y_1'(0)}{k_\beta}\frac{e^{-ik_\beta s}}{k_\beta} \\
&\quad + i\frac{Nr_0W_0}{4\gamma C k_\beta}\left(-i y_1(0) s\sin k_\beta s + i\frac{y_1'(0)}{k_\beta} s\cos k_\beta s + y_1(0) s\cos k_\beta s\right. \\
&\qquad\qquad \left. + \frac{1}{k_\beta} y_1(0)\sin k_\beta s - \frac{1}{k_\beta}\frac{y_1'(0)}{k_\beta}\cos k_\beta s + \frac{1}{k_\beta} y_1'(0) s\sin k_\beta s\right) \\
&= \tilde{y}_2(0)\cos k_\beta s - i\tilde{y}_2(0)\sin k_\beta s + i\frac{Nr_0W_0}{4\gamma C k_\beta}\frac{y_1'(0)}{k_\beta}\frac{e^{-ik_\beta s}}{k_\beta} \\
&\quad + i\frac{Nr_0W_0}{4\gamma C k_\beta}\left[\frac{1}{k_\beta}\left(y_1(0)\sin k_\beta s - \frac{y_1'(0)}{k_\beta}\cos k_\beta s\right) + \left(y_1(0) + i\frac{y_1'(0)}{k_\beta}\right) s\left(\cos k_\beta s - i\sin k_\beta s\right)\right] \\
&= \tilde{y}_2(0) e^{-ik_\beta s} + i\frac{Nr_0W_0}{4\gamma C k_\beta}\left[\frac{1}{k_\beta}\left(y_1(0)\sin k_\beta s - \frac{y_1'(0)}{k_\beta}\cos k_\beta s + \frac{y_1'(0)}{k_\beta} e^{-ik_\beta s}\right) + \tilde{y}_1(0) s e^{-ik_\beta s}\right] \\
&= \tilde{y}_2(0) e^{-ik_\beta s} + i\frac{Nr_0W_0}{4\gamma C k_\beta}\left(\frac{1}{k_\beta}\tilde{y}_1^*(0)\sin k_\beta s + \tilde{y}_1(0) s e^{-ik_\beta s}\right)
\end{aligned}
\tag{24}
$$

与 Eq. (4.40) 一致。